

Online Appendix of “Learning in a Small/Big World”

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A Switching between different automaton

This section illustrates how the setup in this paper encompasses learning mechanisms that involve switching between automaton with size smaller than M .

I first argue that assuming that the DM could switch between multiple learning mechanisms with M memory states implicitly implies that he has a larger memory capacity than M . Suppose that a DM starts with $(\mathcal{T}, d)$ at memory state m_1 and switches to $(\mathcal{T}', d')$ and $(\mathcal{T}'', d'')$ once he transit to memory state m_1 and m_2 respectively. As $(\mathcal{T}, d) \neq (\mathcal{T}', d') \neq (\mathcal{T}'', d'')$, when the DM receives a signal s at memory state m and decides to which memory state he transit to, or when he decides which action he takes at memory state m , he has to remember whether he has once transited to memory state m_1 and m_2 . In other words, the DM has to track not only his current memory states, but also has to memorize the (incomplete) history of his previous memory states. Switching between multiple automaton thus implicitly implies a larger memory capacity.

Now, I illustrate through an example that a M memory states automaton could be designed to involve switching between automaton with smaller sizes. The example is illustrated in Figure A.1. In this example, the DM starts at memory state 3 with a learning mechanism $(\mathcal{T}, d)$ that involves 5 memory states 1 to 5. Once he transits to memory state 1, he switches to another learning mechanism $(\mathcal{T}', d')$ that involves 5 memory states 1, 10, 11, 12, 13. On the other hand, once he transits to memory state 5, he switches to the learning mechanism $(\mathcal{T}'', d'')$ that involves 5 memory states 5, 6, 7, 8, 9. Thus, the proposed 13 memory states automaton can be interpreted as a mechanism that involves switching between three 5 memory states automaton.

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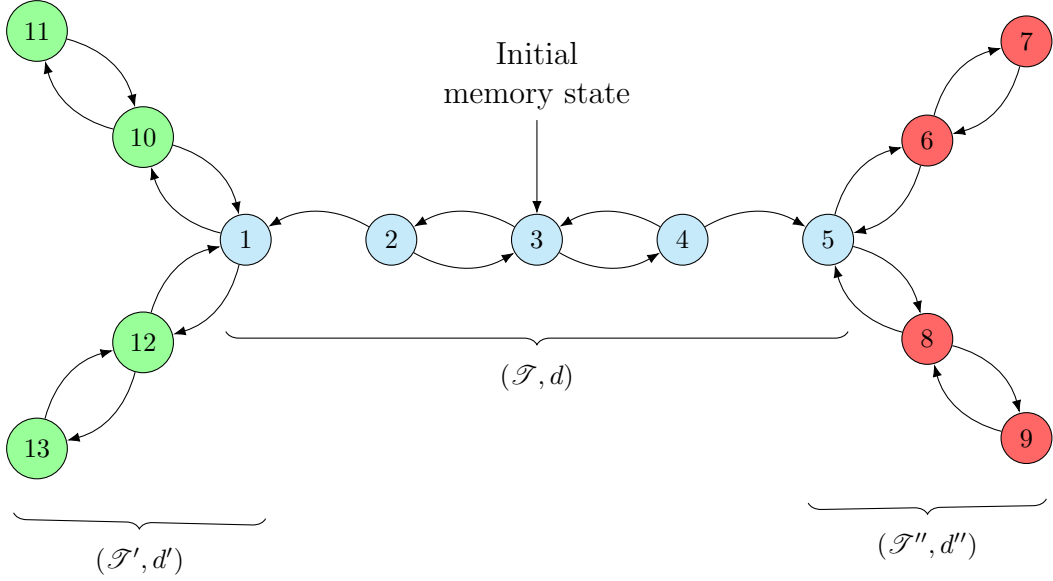


Figure A.1: An example of a 13 memory states automaton that involves switching between three 5 memory states automaton. The DM starts at memory state 3 with $(\mathcal{T}, d)$, and switches to $(\mathcal{T}', d')$ and $(\mathcal{T}'', d'')$ once he transits to memory state 1 and 5, respectively.

B $N, M \rightarrow \infty$: moved to main text

The analysis of the limit $N, M \rightarrow \infty$, including the formal setup, assumptions, propositions, corollaries, and all proofs, has been moved to Sections 5.1.1–5.3.1 of the main paper and the corresponding subsection of the main paper’s Appendix. Please refer to those sections.

C Example of a sequence of signal structures that satisfies Equation (Limit-NRO) and Assumption (Limit-ID)

In this section, I provide an example of a sequence of signal structures that satisfies Equation (Limit-NRO) and Assumption (Limit-ID). Consider the following sequence of signal structures $\{(f_N^\omega)_{\omega=1}^N\}_{N=1}^\infty$ with $S = [0, 1)$:

$$f_N^\omega(s) = \begin{cases} \frac{2}{3} & \text{for } s \in [\frac{i}{2^\omega}, \frac{i+1}{2^\omega}) \text{ where } i = 0, 2, \dots, 2^\omega - 2; \\ \frac{4}{3} & \text{for } s \in [\frac{i}{2^\omega}, \frac{i+1}{2^\omega}) \text{ where } i = 1, 3, \dots, 2^\omega - 1. \end{cases}$$

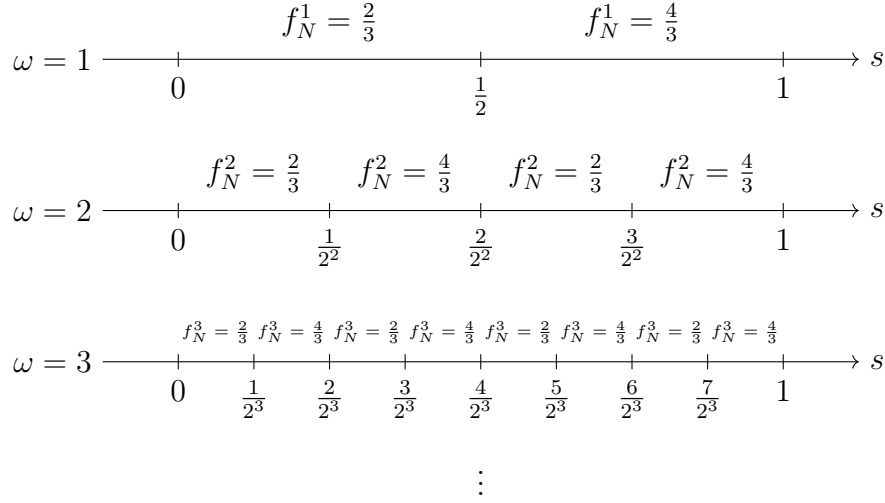


Figure C.2: Signal structures that satisfies Equation (Limit-NRO) and Assumption (Limit-ID) as $N \rightarrow \infty$. The signal structures comprise low and high density alternatively in 2^ω equal-sized intervals.

The signal structures are illustrated in Figure C.2. First, they satisfy Equation (Limit-NRO) as $\frac{f_N^\omega(s)}{f_N^{\omega'}(s)} \geq \frac{1}{2}$ for all $\omega, \omega' \in \Omega$, N and s . Second, for any N and given any ω and $\omega' \neq \omega$, the Cauchy-Schwarz distance is equal to:

$$\begin{aligned}
 -\log \frac{\int f_N^\omega(s) f_N^{\omega'}(s) ds}{\sqrt{\int (f_N^\omega(s))^2 ds} \sqrt{\int (f_N^{\omega'}(s))^2 ds}} &= -\log \frac{\frac{1}{4}(\frac{2}{3})^2 + \frac{1}{2}\frac{2}{3}\frac{4}{3} + \frac{1}{4}(\frac{4}{3})^2}{\frac{1}{2}(\frac{2}{3})^2 + \frac{1}{2}(\frac{4}{3})^2} \\
 &= -\log \frac{1}{10/9} > 1 + \xi
 \end{aligned}$$

where $\xi < \log \frac{10}{9} - 1$ and thus satisfies Assumption (Limit-ID).

D Robustness of the results in small worlds to updating mistakes

Below, I show that the main implications in small worlds, i.e., the asymptotic distribution of actions is close to Bayesian and that disagreement does not persist, hold even when individuals make “updating mistakes”. In other words, the results are robust to individuals’ limited ability to design and follow an “optimal” updating mechanism.

Consider two individuals A and B . Individual A adopts the simple updating mechanism described in the proof of Proposition 1 while individual B “attempts” to adopt the same

updating mechanism but makes local mistakes as he randomly transits to neighboring memory states with some probability $\gamma \in (0, 1)$. Formally, the transition rule of individual B , denoted as $\mathcal{T}'(m, s)$, is as follows:

$$\mathcal{T}'(0, s) = \begin{cases} \mathcal{T}(0, s) & \text{with probability } 1 - \gamma; \\ ji & \text{with probability } \frac{\gamma}{N} \text{ for all } j = 1, \dots, N. \end{cases}$$

$$\mathcal{T}'(i1, s) = \begin{cases} \mathcal{T}(i1, s) & \text{with probability } 1 - \gamma; \\ i2 & \text{with probability } \frac{\gamma}{2}; \\ 0 & \text{with probability } \frac{\gamma}{2}. \end{cases}$$

$$\mathcal{T}'(i\lambda, s) = \begin{cases} \mathcal{T}(i\lambda, s) & \text{with probability } 1 - \gamma; \\ i(\lambda - 1) & \text{with probability } \gamma. \end{cases}$$

while for $k = 2, 3, \dots, \lambda - 1$,

$$\mathcal{T}'(ik, s) = \begin{cases} \mathcal{T}(ik, s) & \text{with probability } 1 - \gamma; \\ i(k - 1) & \text{with probability } \frac{\gamma}{2}; \\ i(k + 1) & \text{with probability } \frac{\gamma}{2}. \end{cases}$$

where $\mathcal{T}(m, s)$ is defined in the proof of Proposition 1. The initial memory states and decision rule of individual B stay the same as individual A . Such updating mistakes could be induced by memory imperfections, i.e., the DM's memory state is subject to local fluctuations, or imperfect perceptions on signals, e.g., the DM may mistakenly perceive any signal as a signal that supports state ω .

Proposition D.1. *Consider individual A who adopts the updating mechanism $\mathcal{T}$ and individual B who makes local mistakes with some probability $\gamma \in (0, 1)$, characterized above by $\mathcal{T}'(m, s)$. Fix N and $(\mathbf{u}_B^\omega, p_B^\omega, f_B^\omega)_{\omega=1}^N$, for all $\gamma \in (0, 1)$,*

$$\lim_{k \rightarrow \infty} L_k(m_1, \mathcal{T}', d) = 0.$$

Moreover, individual A and B are bound to agree in small worlds.

E Proof of the results in online appendix

E.1 Proof of Proposition D.1

Proof. The proof follows closely the proof of Proposition 1. For individual B , fixing the state ω , in the stationary probability distribution, we have at the two extreme memory states in branch ω' ,

$$\mu_{\omega'\lambda}^\omega \left[\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(G^{\omega''}) + \gamma \right] = \mu_{\omega'(\lambda-1)}^\omega \left[(1-\gamma)F^\omega(G^{\omega'}) + \frac{\gamma}{2} \right]$$

$$\mu_{\omega'(\lambda-1)}^\omega = \mu_{\omega'\lambda}^\omega \left[\frac{(1-\gamma)F^\omega(G^{\omega'}) + \frac{\gamma}{2}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(G^{\omega''}) + \gamma} \right]^{-1}$$

for all ω' . Similarly, at memory state $\omega'(\lambda-1)$,

$$\mu_{\omega'\lambda}^\omega \left[\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(S^{\omega''}) + \gamma \right] + \mu_{\omega'(\lambda-2)}^\omega \left[(1-\gamma)F^\omega(S^{\omega'}) + \frac{\gamma}{2} \right] = \mu_{\omega'(\lambda-1)}^\omega \left[\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(S^{\omega''}) + (1-\gamma)F^\omega(S^{\omega'}) + \gamma \right]$$

$$\mu_{\omega'(\lambda-2)}^\omega \left[(1-\gamma)F^\omega(S^{\omega'}) + \frac{\gamma}{2} \right] = \mu_{\omega'(\lambda-1)}^\omega \left[\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(S^{\omega''}) + \frac{\gamma}{2} \right]$$

$$\mu_{\omega'(\lambda-2)}^\omega = \mu_{\omega'(\lambda-1)}^\omega \left[\frac{(1-\gamma)F^\omega(S^{\omega'}) + \frac{\gamma}{2}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(S^{\omega''}) + \frac{\gamma}{2}} \right]^{-1}$$

Repeating the same procedures implies that for all $k = 1, \dots, \lambda-1$

$$\mu_{\omega'k}^\omega = \mu_{\omega'\lambda}^\omega \left[\frac{(1-\gamma)F^\omega(S^{\omega'}) + \frac{\gamma}{2}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(S^{\omega''}) + \gamma} \right]^{-1} \left[\frac{(1-\gamma)F^\omega(S^{\omega'}) + \frac{\gamma}{2}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(S^{\omega''}) + \frac{\gamma}{2}} \right]^{-(\lambda-k-1)} \quad (\text{E.1})$$

and

$$\mu_0^\omega = \mu_{\omega'\lambda}^\omega \left[\frac{(1-\gamma)F^\omega(S^{\omega'}) + \frac{\gamma}{2}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(S^{\omega''}) + \gamma} \right]^{-1} \left[\frac{(1-\gamma)F^\omega(S^{\omega'}) + \frac{\gamma}{2}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(S^{\omega''}) + \frac{\gamma}{2}} \right]^{-(\lambda-2)} \left[\frac{(1-\gamma)F^\omega(S^{\omega'}) + \frac{\gamma}{N}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega'} F^\omega(S^{\omega''}) + \frac{\gamma}{2}} \right] \quad (\text{E.2})$$

Note that for all $\gamma \in (0, 1)$

$$\frac{(1-\gamma)F^\omega(S^\omega) + \frac{\gamma}{2}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega} F^\omega(S^{\omega''}) + \frac{\gamma}{2}} > 1.$$

$$\frac{(1-\gamma)F^\omega(S^\omega) + \frac{\gamma}{2}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega} F^\omega(S^{\omega''}) + \frac{\gamma}{2}} > \frac{(1-\gamma)F^\omega(S^{\omega'}) + \frac{\gamma}{2}}{\frac{1-\gamma}{\delta} \sum_{\omega'' \neq \omega} F^\omega(S^{\omega''}) + \frac{\gamma}{2}}.$$

Then the result follows from the same steps in the proof of Proposition [1](#).

□